数理解析研究所講究録
1511 巻 2006 年 136-144

A variant of the Bombieri-Vinogradov theorem

By

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Abstract. In the present article we establish a variant of the Bombieri-Vinogradov theorem for primes in arithmetic progressions to widely separated moduli. Our result is derived by the circle method, Linnik's dispersion method, and an application of the Burgess's bound for character sums via Iwaniec's version of the Brun-Titchmarsh theorem [4].

1. Introduction. Le Λ and φ denote the von Mangoldt function and the Euler function, respectively. Consider the error term in the prime number theorem for arithmetic progressions

$$E(x;q,a)=\sum_{\substack{n\le x\\ n\equiv a\,(\mathrm{mod}\,q)}}\Lambda(n)-\frac{x}{\varphi(q)},$$

where $(a,q)=1$. The famous Siegel–Walfisz theorem states that for arbitrary constants $A,B>0$,

$$|E(x;q,a)|\ll\frac{x}{q}(\log x)^{-A}, \tag{1}$$

uniformly for $q\le(\log x)^B$ and $(a,q)=1$. It is worth noting that if the zero-free region for the Dirichlet L-functions were "wide" enough then Huxley's zero-density estimate would yield (1) for $q\le x^{5/12-\varepsilon}$, $\varepsilon>0$. Furthermore, if the generalized Riemann hypothesis were valid then (1) would hold for $q\le x^{1/2}(\log x)^{-A-2}$.

Another well-known result in analytic number theory, the Bombieri–Vinogradov theorem, has the same strength as the generalized Riemann hypothesis on average, and therefore enables us to deal with various problems. More specifically, it asserts that for every $A>0$ there exists a constant $B=B(A)>0$ such that

$$\sum_{q\le x^{1/2}(\log x)^{-B}}\max_{(a,q)=1}|E(x;q,a)|\ll\frac{x}{(\log x)^A},$$

see [1], for example.

Not long ago, Elliott [2], [3] investigated the solvability of polynomial equations in primes and demonstrated a surprising result concerning the Bombieri–Vinogradov theorem. Given a polynomial f of degree greater than 1, with integer coefficients, leading coefficient positive, Elliott showed that for any $A>0$,

$$\sum_{\substack{k\\ 0<f(k)\le x^{\theta}}}\frac{\varphi(f(k))}{k}\max_{(a,f(k))=1}|E(x;f(k),a)|\ll\frac{x}{(\log x)^A}, \tag{2}$$

Mathematics Subject Classification (2000): 11N13, 11P55.

1) The author was supported by the Japan Society for the Promotion of Science, P03763.

provided that $0 \le \theta < 1/4$. Notice that the left hand side is an average only over a sparse set of moduli. The role of the factor $\varphi(f(k))/k$ is discussed in [2] in detail.

It would be of some interest to establish (2) for wider ranges of the moduli. In the present paper we prove the following result.

Theorem 1. *Let $A > 0$, $\varepsilon > 0$ be given. Then the inequality (2) holds, provided that $0 \le \theta \le 8/19 - \varepsilon$. The implied constant depends on A, ε and the polynomial f.*

The exponent 8/19 comes from Burgess's bound for character sums via Iwaniec's version of the Brun–Titchmarsh theorem [4], see the end of section 2 of the present paper. Also, for comparison, 8/19 is a little larger than 5/12.

In order to prove theorem 1, we apply Vaughan's identity, and reduce the problem to the estimation of the so-called type I and type II sums. We treat type I sums by appealing to the above mentioned estimate of Iwaniec. To deal with type II sums, we employ Linnik's dispersion method, as in Elliott [2]. We are thus required to prove the following Barban–Davenport–Halberstam type inequality.

Theorem 2. *Let $A > 0$ be given. Then there exists a constant $B > 0$, depending on A and the polynomial f, such that*

$$\sum_{\substack{k \\ 0<f(k)\le x(\log x)^{-B}}} \frac{\varphi(f(k))}{k} \sum_{\substack{1\le a\le f(k) \\ (a,f(k))=1}} (E(x;f(k),a))^2 \ll \frac{x^2}{(\log x)^A},$$

where the implied constant depends on A and the polynomial f.

In order to deduce theorem 2, we first adopt Montgomery's approach [5], and then apply the circle method in a classical way. For this reason, the fact that f is a a polynomial is essential to our argument, and enables us to bound the exponential sums involving f. A possible value for B is given at the end of section 3.

Notation. Throughout, d denotes the degree of the polynomial f, x is a sufficiently large number, $L = \log x$. For real t, we write $\|t\|$ for the distance from t to the nearest integer, $e(t) = \exp(2\pi i t)$. Instead of $m \equiv n \pmod q$ we write for simplicity $m \equiv n\ (q)$. As usual, $\mu(n)$ and $\tau_k(n)$ denote the Möbius function and the k-th divisor function, respectively. The notation $n \sim X$ means that n runs through a *sub-interval* of $(X, 2X]$, whose endpoints are not necessarily the same in the different equations and may depend on the outer summation variables.

2. Proof of Theorem 1. We begin by decomposing Λ via Vaughan's identity [1, §24] with the parameter Z to be chosen later. We have

$$\Lambda(n) = \Lambda_1(n) - \Lambda_2(n) + \Lambda_3(n) - \Lambda_4(n),$$

where

$$\Lambda_1(n) = \begin{cases} \Lambda(n), & \text{if } n \le Z, \\ 0, & \text{otherwise,} \end{cases}$$

$$\Lambda_2(n) = \sum\sum_{\substack{jhr=n \\ j,h\le Z}}\sum \Lambda(j)\mu(h), \quad \Lambda_3(n) = \sum_{\substack{jh=n \\ h\le Z}}\sum (\log j)\,\mu(h),$$

and

$$\Lambda_4(n) = \sum_{\substack{jh=n \\ j,h>Z}}\sum \Lambda(j)g(h); \qquad |g(h)| \ll \tau(h).$$

Denote $Q = x^{\theta}$. Then, by the prime number theorem, it is sufficient to prove that for $1 \le i \le 4$,

$$G_i := \sum_k{}^{\dagger} \frac{\varphi(f(k))}{k} \max_{(a,f(k))=1} |E_i(x; f(k), a)| \ll xL^{-A}, \tag{3}$$

where $\dagger$ in $\sum_k^{\dagger}$ stands for the condition $0 < f(k) \le Q$, and

$$E_i(x; f(k), a) = \sum_{\substack{n\le x \\ n\equiv a\,(f(k))}} \Lambda_i(n) - \frac{1}{\varphi(f(k))} \sum_{\substack{n\le x \\ (n,f(k))=1}} \Lambda_i(n).$$

2.1. Type II Sums. We first consider G_4, so as to determine the parameter Z. By a dyadic decomposition of the summation ranges and Cauchy's inequality, we see that

$$\begin{aligned}(G_4)^2 \ll\ & L^4 \max_{\substack{M,N\ge Z \\ MN\le x}} \sum_{k'}{}^{\dagger} \frac{1}{k'} \sum_{m'\sim M} |g(m')|^2 \\ & \times \sum_k{}^{\dagger} \frac{\varphi(f(k))^2}{k} \max_{(a,f(k))=1} \mathcal{D}(M, N; f(k), a), \end{aligned} \tag{4}$$

where

$$\mathcal{D}(M, N; f(k), a) = \sum_{\substack{m\sim M \\ (m,f(k))=1}} \left| \sum_{\substack{n\sim N \\ mn\le x \\ mn\equiv a\,(f(k))}} \Lambda(n) - \frac{1}{\varphi(f(k))} \sum_{\substack{n\sim N \\ mn\le x \\ (n,f(k))=1}} \Lambda(n) \right|^2.$$

Expanding the square out, we have

$$\mathcal{D} = W - 2V + U,$$

with the obvious notation.

We proceed to deal with

$$W = \sum_{\substack{m\sim M \\ (m,f(k))=1}} \sum_{\substack{n,n'\sim N \\ mn,mn'\le x \\ mn\equiv a\,(f(k)) \\ mn'\equiv a\,(f(k))}}\sum \Lambda(n)\Lambda(n').$$

We bring the sum over m inside. Since $(a, f(k)) = 1$, the system of congruences is soluble if and only if $(nn', f(k)) = 1$ and $n \equiv n'\,(f(k))$, and reduces to $m \equiv \overline{n}a\,(f(k))$, which implies $(m, f(k)) = 1$. Then, by the elementary evaluation

$$\sum_{\substack{Y<m\le X \\ m\equiv c\,(q)}} 1 = \frac{1}{q} \sum_{Y<m\le X} 1 + \mathcal{O}(1),$$

we have that

$$
\begin{aligned}
W &= \sum_{\substack{n,n'\sim N\\ (nn',f(k))=1\\ n\equiv n'\,(f(k))}}\sum \Lambda(n)\Lambda(n')\left(\frac{1}{f(k)}\sum_{\substack{m\sim M\\ mn,mn'\le x}} 1+\mathcal{O}(1)\right)\\
&= \frac{1}{f(k)}\sum_{m\sim M}\sum_{\substack{1\le b\le f(k)\\ (b,f(k))=1}}\left(\sum_{\substack{n\sim N\\ mn\le x\\ n\equiv b\,(f(k))}}\Lambda(n)\right)^2+\mathcal{O}\left(NL\left(\frac{N}{f(k)}+1\right)\right).
\end{aligned}
$$

Similarly,

$$
U,V=\frac{1}{f(k)}\sum_{m\sim M}\frac{1}{\varphi(f(k))}\left(\sum_{\substack{n\sim N\\ mn\le x\\ (n,f(k))=1}}\Lambda(n)\right)^2+\mathcal{O}\left(\frac{N^2}{\varphi(f(k))}\right).
$$

Therefore, uniformly for $(a,f(k))=1$ we have that

$$
\begin{aligned}
\mathcal{D} &= \frac{1}{f(k)}\sum_{m\sim M}\sum_{\substack{1\le b\le f(k)\\ (b,f(k))=1}}\left|\sum_{\substack{n\sim N\\ mn\le x\\ n\equiv b\,(f(k))}}\Lambda(n)-\frac{1}{\varphi(f(k))}\sum_{\substack{n\sim N\\ mn\le x\\ (n,f(k))=1}}\Lambda(n)\right|^2\\
&\quad+\mathcal{O}\left(NL\left(\frac{N}{f(k)}+1\right)\right)\\
&= \frac{1}{f(k)}\sum_{m\sim M}\sum_{\substack{1\le b\le f(k)\\ (b,f(k))=1}}|E(N,m;f(k),b)|^2+\mathcal{O}\left(NL\left(\frac{N}{f(k)}+1\right)\right),
\end{aligned}
$$

say. Substituting this into (4), we obtain that

$$
\begin{aligned}
(G_4)^2 &\ll L^8\max_{\substack{M,N\ge Z\\ MN\le x}} M\left(\sum_k{}^{\dagger}\frac{\varphi(f(k))^2}{k\,f(k)}\sum_m\sum_b|E|^2+NL^2(QN+Q^2)\right)\\
&\ll L^8\max_{\substack{M,N\ge Z\\ MN\le x}} M\sum_{m\sim M}\sum_{\substack{k\\ 0<f(k)\le Q}}\frac{\varphi(f(k))}{k}\sum_{\substack{1\le b\le f(k)\\ (b,f(k))=1}}|E(N,m;f(k),b)|^2\\
&\quad+xL^{10}\left(\frac{Qx}{Z}+Q^2\right).
\end{aligned}
$$

We now take $Z=Qx^{\varepsilon}=x^{\theta+\varepsilon}\le x^{8/19}$, so that $Q=Zx^{-\varepsilon}\le Nx^{-\varepsilon}$, and $Qx/Z=x^{1-\varepsilon}$. Then theorem 2 yields the desired bound for G_4.

2.2. Type I Sums. For $1 \le i \le 3$, we shall show that

$$E_i(x; q, a) \ll \frac{x}{q} L^{-A-1},$$

uniformly for $q \le Q$ and $(a, q) = 1$, not averaging over moduli $q\ (= f(k))$. Then this assertion gives (3).

The contribution of E_1 is negligible. By partial summation and an elementary estimate, we find that

$$E_3 \ll ZL,$$

which is also acceptable, since $qZ \le QZ = x^{2\theta+\varepsilon} \le x^{1-\varepsilon}$.

To estimate E_2, we appeal to Iwaniec's result [4, Theorem 3]. Then, assuming that $Q \le x^{9/20-\eta}$ and $Z^2 \le x^{1-\eta} Q^{-3/8}$, $\eta > 0$, we have that

$$E_2 = \sum_{\substack{m,n \le Z \\ (mn,q)=1}} \sum \Lambda(m)\mu(n) \left(\sum_{\substack{l \le x/mn \\ lmn \equiv a\,(q)}} 1 - \frac{x}{qmn} \right) + \mathcal{O}\left(Z^2 L \frac{\tau(q)}{\varphi(q)} \right) \ll \frac{x}{q} L^{-A-1}.$$

In fact, the above assumptions are fulfilled, since $Z = Qx^{\varepsilon} = x^{\theta+\varepsilon} \le x^{8/19}$, and $Z^2 Q^{3/8} = Q^{19/8} x^{2\varepsilon} \le x^{1-3\varepsilon/8}$. Our proof of theorem 1 is thus complete.

3. Proof of Theorem 2. For small $f(k)$'s we apply the Siegel–Walfisz theorem. Then the average under consideration becomes

$$\sum_k{}' \frac{\varphi(f(k))}{k} \sum_{\substack{1 \le a \le f(k) \\ (a, f(k))=1}} (E(x; f(k), a))^2 + \mathcal{O}(x^2 L^{-A}),$$

where $'$ in $\sum_k'$ denotes the restriction $L^C \le f(k) \le xL^{-B}$, with an arbitrary constant $C > 0$ to be specified later. As in [5, Chapter 17], by the prime number theorem, the above is

$$\begin{aligned}
&= 2 \sum_k{}' \sum_m \frac{\varphi(f(k))}{k} \sum_{\substack{n,n' \le x \\ n-n'=f(k)m}} \sum \Lambda(n)\Lambda(n') - x^2 \sum_k{}' \frac{1}{k} + \mathcal{O}(x^2 L^{-A}) \\
&= 2R - x^2 \sum_k{}' \frac{1}{k} + \mathcal{O}(x^2 L^{-A}), \qquad (5)
\end{aligned}$$

say. To evaluate R, we use the circle method in a standard way, see [6, Chapter 3]. We have

$$R = \int_0^1 |S(\alpha)|^2 F(\alpha) d\alpha,$$

where

$$S(\alpha) = \sum_{n \le x} \Lambda(n)\, e(\alpha n); \quad F(\alpha) = \sum_{\substack{k \quad m \\ f(k)m \le x}}{}' \sum \frac{\varphi(f(k))}{k}\, e(\alpha f(k) m).$$

We decompose the unit interval by Farey dissection of order Q, and define the major arcs $\mathfrak{M}$ as the union of all intervals $\{\alpha \in \mathbb{R} : |q\alpha - a| \le Q^{-1}\}$ with $1 \le a \le q \le P$ and $(a,q) = 1$. We take

$$P = L^D; \quad PQ = xL^{-A-2},$$

where the constant $D > 0$ will be determined later. Put $I = [1/Q, 1+1/Q]$, and define $\mathfrak{m} = I \backslash \mathfrak{M}$ to be the minor arcs. Then we see that

$$R = \left(\int_{\mathfrak{M}} + \int_{\mathfrak{m}}\right) |S(\alpha)|^2 F(\alpha) d\alpha = R(\mathfrak{M}) + R(\mathfrak{m}),$$

say.

3.1. The major arcs. Since $F(\alpha) \ll xL$, the standard argument based on the Siegel–Walfisz theorem yields

$$R(\mathfrak{M}) = \sum_{q \le P} \sum_{\substack{1 \le a \le q \\ (a,q)=1}} \int_{-1/2}^{1/2} \left| \frac{\mu(q)}{\varphi(q)} T(\beta) \right|^2 F\left(\frac{a}{q} + \beta\right) d\beta + O(x^2 L^{-A}),$$

where $T(\alpha) = \sum_{n \le x} e(\alpha n)$. After integrating and summing over a, the main expression becomes

$$\begin{aligned} &= \sum_{q \le P} \frac{\mu(q)^2}{\varphi(q)^2} \sideset{}{'}\sum_k \sum_{\substack{m \\ f(k)m \le x}} \frac{\varphi(f(k))}{k} c_q(f(k)m) \sum_{\substack{n,n' \le x \\ n-n'=f(k)m}} 1 \\ &= \sum_{q \le P} \frac{\mu(q)^2}{\varphi(q)^2} \sideset{}{'}\sum_k \sum_{\substack{m \\ f(k)m \le x}} \frac{\varphi(f(k))}{k} c_q(f(k)m)\,(x - f(k)m) + O(xL^2). \qquad (6) \end{aligned}$$

From the definition of the Ramanujan sum $c_q(n)$ we have

$$\begin{aligned} \sum_{m \le x/f(k)} c_q(f(k)m)(x - f(k)m) &= \sum_{\substack{1 \le c \le q \\ (c,q)=1}} \sum_{1 \le b \le q} e\left(\frac{c}{q} f(k) b\right) \sum_{\substack{m \le x/f(k) \\ m \equiv b\ (q)}} (x - f(k)m) \\ &= \frac{x^2}{2f(k)q} \sum_{\substack{1 \le c \le q \\ (c,q)=1}} \sum_{1 \le b \le q} e\left(\frac{c}{q} f(k) b\right) + O(x\varphi(q)q) \\ &= \frac{x^2}{2f(k)} \varphi(q) + O(x\varphi(q)q), \end{aligned}$$

where the main term vanishes unless $q | f(k)$. Whence (6) is

$$= \frac{x^2}{2} \sideset{}{'}\sum_k \frac{\varphi(f(k))}{kf(k)} \sum_{\substack{q \le P \\ q | f(k)}} \frac{\mu(q)^2}{\varphi(q)} + O(x^2 PL^{1-B}) = R_0 + O(x^2 PL^{1-B}),$$

say. Thus, on taking $B = A + D + 1$, we infer that

$$R(\mathfrak{M}) = R_0 + O(x^2 L^{-A}). \qquad (7)$$

Note that

$$R_0 \leq \frac{x^2}{2}\sum_k{}' \frac{1}{k}, \tag{8}$$

since

$$\sum_{\substack{q \leq P \\ q | f(k)}} \frac{\mu(q)^2}{\varphi(q)} \leq \sum_{q | f(k)} \frac{\mu(q)^2}{\varphi(q)} = \frac{f(k)}{\varphi(f(k))}.$$

3.2 The minor arcs. Since

$$R(\mathfrak{m}) \ll xL \sup_{\alpha \in \mathfrak{m}} |F(\alpha)|,$$

our objective is to bound the exponential sum $F(\alpha)$. For any $\alpha \in \mathfrak{m}$, there exists a rational number a/q such that $|\alpha - a/q| \leq 1/q^2$, $(a,q) = 1$ and $P \leq q \leq Q$, by the Dirichlet approximation theorem.

We first manage the factor $\varphi(f(k))/k$. We have

$$\begin{aligned} |F(\alpha)| &\leq \sum_k{}' \frac{\varphi(f(k))}{k} \left| \sum_{m \leq x/f(k)} e(\alpha f(k) m) \right| \\ &\ll \sum_k{}' \min\left(\frac{x}{k}, \frac{f(k)}{k\|\alpha f(k)\|}\right) \ll \sum_k{}' \min\left(\frac{x}{k}, \frac{k^{d-1}}{\|\alpha f(k)\|}\right), \end{aligned}$$

recalling that f is of degree d. By a dyadic decomposition of the summation range for k, we obtain that

$$F(\alpha) \ll L \max_K K^{d-1} \sum_{k \sim K} \min\left(\frac{x}{K^d}, \frac{1}{\|\alpha f(k)\|}\right), \tag{9}$$

where the maximum is taken over K satisfying $L^C \ll K^d \ll xL^{-B}$. Put $M = xK^{-d} \gg L^B$. By the Fourier expansion

$$\min(M, \|\xi\|^{-1}) = \sum_{m=-\infty}^{\infty} w_m e(\xi m); \qquad |w_m| \ll \min\left(\log M, \frac{M}{|m|}, \frac{M^2}{m^2}\right),$$

we find that

$$\begin{aligned} \sum_{k \sim K} \min\left(M, \|\alpha f(k)\|^{-1}\right) &\ll KL + \sum_{m \leq M^2} \min\left(L, \frac{M}{m}\right) \left| \sum_{k \sim K} e(\alpha f(k) m) \right| \\ &\ll KL + L \max_{1 \leq T \leq M} \frac{1}{T} \sum_{m \leq MT} \left| \sum_{k \sim K} e(\alpha f(k) m) \right|. \end{aligned} \tag{10}$$

We are ready to apply Weyl's shift [6, Lemma 2.3] to the above exponential sum. On putting $H = 2^{d-1}$, we have that

$$\begin{aligned} &\frac{1}{T}\sum_{m\le MT}\left|\sum_{k\sim K} e(\alpha f(k)m)\right|^H \\ &\qquad \ll \; MK^{H-1} + T^{-1}\sum_{h\ll MTK^{H-1}} \tau_d(h)\min\left(K, \|\alpha h\|^{-1}\right). \end{aligned} \tag{11}$$

By Cauchy's inequality and [6, Lemma 2.2], the sum over h is

$$\begin{aligned} &\ll \left(K\sum_{h\ll MTK^{H-1}} \tau_d(h)^2\right)^{1/2}\left(\sum_{h\ll MTK^{H-1}} \min\left(K, \|\alpha h\|^{-1}\right)\right)^{1/2} \\ &\ll \left(MTK^H L^{d^2-1}\right)^{1/2}\left(L\left(\frac{MTK^H}{q} + MTK^{H-1} + q\right)\right)^{1/2} \\ &\ll L^{d^2/2}\left(\frac{MTK^H}{q^{1/2}} + \frac{MTK^H}{K^{1/2}} + (qMTK^H)^{1/2}\right) \\ &\ll TK^{H-d}L^{d^2/2}\left(\frac{x}{q^{1/2}} + \frac{x}{K^{1/2}} + (qx)^{1/2}\right) \\ &= TK^{H-d}L^{d^2/2}J(K), \end{aligned}$$

say, since $MK^d = x$, $H = 2^{d-1} \ge d$, and $T \ge 1$. Then the right hand side of (11) is bounded by

$$\ll K^{H-d}L^{d^2/2}J(K).$$

Combining this with (10) by means of Hölder's inequality, we obtain that

$$\begin{aligned} \sum_{k\sim K}\min(M, \|\alpha f(k)\|^{-1}) &\ll KL + LM^{1-1/H}K^{1-d/H}L^{d^2/2H}J(K)^{1/H} \\ &\ll KL + MK(MK^d)^{-1/H}L^3 J(K)^{1/H}. \end{aligned}$$

It turns out, by (9), that

$$\begin{aligned} F(\alpha) &\ll \max_K \left(K^d L^2 + (MK^d)^{1-1/H}L^4 J(K)^{1/H}\right) \\ &\ll xL^{2-B} + x^{1-1/H}L^4 J(L^{C/d})^{1/H}. \end{aligned}$$

Now, for $\alpha \in \mathfrak{m}$, we find that

$$J(L^{C/d}) \ll \frac{x}{P^{1/2}} + \frac{x}{L^{C/(2d)}} + (Qx)^{1/2} \ll \frac{x}{L^{D/2}} + \frac{x}{L^{C/(2d)}}.$$

On choosing $C = 2^d d(A+5)$ and $D = 2^d(A+5)$, we conclude that

$$\sup_{\alpha\in\mathfrak{m}} F(\alpha) \ll xL^{-A-1},$$

or

$$R(\mathfrak{m}) \ll x^2 L^{-A}. \tag{12}$$

Note that $B = 2^d(A+5) + A + 1$.

Finally, putting together (7), (8) and (12), we find that

$$R = R_0 + \mathcal{O}(x^2L^{-A}) \leq \frac{x^2}{2}\sum_k{}' \frac{1}{k} + \mathcal{O}(x^2L^{-A}) .$$

Theorem 2 follows from this and (5).

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